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A NOTE ON THE CLASS OF N-POWER QUASI-ISOMETRY

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ABSTRACT

In this paper we introduce a new class of operators called the n - power quasi - isometry and study their properties related to quasinormality and partial isometry. We also introduce another related new class of operators and investigate their spectral properties.

KEYWORDS: Hilbert Space, Isometry, Operator, Quasinormal, Quasi-Isometry

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1. INTRODUCTION

Let T be a bounded linear operator on a complex Hilbert space H. T is said to be

(i) Normal if $T^*T = TT^*$

(ii) N-Normal if $T^*T^n = T^nT^*$ [2]

(iii) Quasinormal if $T(T^*T) = (T^*T)T$

(iv) Quasi - Isometry if $T^{*2}T^2 = T^*T$

(v) N-Power Quasinormal if $T^*TT^n = T^nT^*T$ [6].

The class of normal, n-normal, quasinormal and n-power quasinormal operators are denoted by [N], [nN], [QN] and [nQN] respectively. The class of quasi-isometries which is a simple extension of isometries was introduced by[4]. The quasi-isometry operators retain some properties of isometries[5]. We introduce a new class of operators T namely n-power quasi-isometry denoted by [nQI] satisfying $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$, $n \in N$. Obviously this is based on the class of quasi-isometries denoted by [QI][4]. It is evident that when n = 1, [1QI] = [QI]. Interestingly we observe that, for n = 1, 2, 3,... the corresponding classes [nQI] are independent which is evident from the following examples.

Example 1.1: For the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, a simple calculation shows that $T \notin [QI]$ but $T \in [2QI]$.

Example 1.2: The operator $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, is [QI] but not [2QI].

Example 1.3: When $H = \ell^2$, the unilateral shift given by the matrix operator $T = \begin{pmatrix} 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . & . \\ \dots & \dots & \dots & . & . & . \end{pmatrix}$, satisfies $T(T^{*2}T^2) = (T^*T)T$ but not $T^2(T^{*2}T^2) = (T^*T)T^2$. That is, $T \in [2QI]$ and $T \notin [3QI]$.

Let $B(H)$ denote the Banach algebra of all bounded linear operators on a complex Hilbert space H and let $\sigma(T)$, $\sigma_a(T)$ and $\sigma_p(T)$ denote the spectrum, the approximate point spectrum and the point spectrum of T respectively.

2. PROPERTIES OF CLASS [nQI]

Definition 2.1: An operator T is said to be unitarily equivalent to an operator S if $S = UTU^*$ for an unitary operator U.

Theorem 2.2: The following assertions hold:

(1) If $T \in [QI] \cap [2QI]$ then $T \in [nQI], n > 2$.

(2) If $T \in [2QI] \cap [3QI]$ then $T \in [nQI], n \geq 4$.

(3) If $T \in [nQI]$ then every operator unitarily equivalent to T is also [nQI].

(4) $T \in [QI] \cap [nQI]$ then $T \in [(n-1)QN], n \geq 2$.

Proof: (1) Since $T \in [QI] \cap [2QI]$, we have,

$$T^{*2}T^2 = T^*T \tag{2.1}$$

$$TT^{*2}T^2 = T^*TT. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain,

$$TT^{*2}T^2 = T^{*2}T^2T \tag{2.3}$$

By (2.1) and (2.3), $(T^*T)T^{n-1} = (T^{*2}T^2)T^{n-1} = (T^{*2}T^2T)T^{n-2} = (TT^{*2}T^2)T^{n-2}$.

Again applying (2.3) in $(T^*T)T^{n-1} = T(T^{*2}T^2T)T^{n-3}$, we obtain $(T^*T)T^{n-1} = T(TT^{*2}T^2)T^{n-3}$.

Repeating the procedure we arrive at $T^*TT^{n-1} = T^{n-1}T^{*2}T^2$.

(2) Since $T \in [2QI] \cap [3QI]$, $T^2T^{*2}T^2 = T^*TT^2$ and $TT^{*2}T^2 = T^*TT$. (2.4)

Combining these two equations we obtain, $T(TT^{*2}T^2) = (TT^{*2}T^2)T$ (2.5)

Using (2.4) and (2.5) we have

$$T^*TT^{n-1} = (T^*TT)T^{n-2} = (TT^{*2}T^2)T^{n-2} = (TT^{*2}T^2)TT^{n-3} = T(TT^{*2}T^2)T^{n-3} = T^2T^{*2}T^2T^{n-3}.$$

Using (2.5) repeatedly for a finite number of times we obtain $T^*TT^{n-1} = T^{n-1}T^{*2}T^2$.

(3) Let S be unitarily equivalent to $T \in [nQI]$. Then $S = UTU^*$, where U is unitary. Since $T \in [nQI]$, $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$ and

$$S^{n-1}S^{*2}S^2 = (U\,T^{n-1}U^*)(UT^{*2}U^*)(UT^2U^*) = U\,(T^{n-1}T^{*2}T^2)U^* = U(T^*T^n)U^* = S^*S^n.$$

(4) Since $T \in [QI] \cap [nQI]$, we have,

$$T^{*2}T^2 = T^*T \tag{2.6}$$

$$T^{n-1}T^{*2}T^2 = T^*TT^{n-1}. \tag{2.7}$$

$$T^*TT^{n-1} = T^{n-1}T^{*2}T^2 \text{ by}(2.7)$$

$$T^*TT^{n-1} = T^{n-1}T^*T \text{ by}(2.6).$$

Hence $T \in [(n-1)QN]$.

It is natural to ask whether the product and sum of two $[nQI]$operators are $[nQI]$. In general they need not be. The following Theorem gives an affirmative answer under some conditions.

Theorem 2.4

(1) If T and S are of class $[nQI]$, such that T doubly commutes with S then $TS \in [nQI]$.

(2) If T and S are of class $[nQI]$, such that $ST = TS = T^*S = ST^* = 0$ then $T + S \in [nQI]$.

Proof: (1) Since T doubly commutes with S, $TS = ST$ and $TS^* = S^*T$.

$(TS)^{n-1}(TS)^{*2}(TS)^2 = T^{n-1}S^{n-1}S^{*2}T^{*2}T^2S^2 = S^{n-1}S^{*2}S^2T^{n-1}T^{*2}T^2 = S^*SS^{n-1}T^*TT^{n-1} = (TS)^*(TS)(TS)^{n-1}$ since $T,S \in [nQI]$.Thus $TS \in [nQI]$.

(2) Since $TS = ST = T^*S = ST^* = 0$, we have,

$$(T+S)^{n-1}(T+S)^{*2}(T+S)^2 = (T^{n-1} + S^{n-1})(T^{*2} + S^{*2})(T^2 + S^2) = T^{n-1}T^{*2}T^2 + S^{n-1}S^{*2}S^2$$

$= T^*TT^{n-1} + S^*SS^{n-1}$ since $T,S \in [nQI]$. $(T+S)^{n-1}(T+S)^{*2}(T+S)^2 = (T+S)^*(T+S)(T+S)^{n-1}$

Thus $T + S \in [nQI]$.

3. CONDITIONS IMPLYING QUASINORMALITY

The class of normal operators and quasinormal operators are independent of class $[nQI]$. In this section we prove that under some algebraic conditions T, T^2 or T^{n-1} are quasinormal.

Theorem 3.1

(1) Let $T \in [QI] \cap [3QI]$then T^2 is quasinormal.

(2) If $T \in [2QI] \cap [3QI]$and $\ker(T^*) \subset \ker(T)$ then is T quasinormal and in particular if $\ker(T^*) = 0$ then T is normal where $\ker T$ is the nullspace of T.

Proof: (1) Since $T \in [QI] \cap [3QI]$, $T^{*2}T^2 = T^*T$ and $T^2T^{*2}T^2 = T^*TT^2$ Hence $T^2T^{*2}T^2 = T^*TT^2 = T^{*2}T^2T^2$ Hence T^2 is quasinormal.

(2) By hypothesis, $TT^{*2}T^2 = T^*TT$ (3.1)

and $T^2T^{*2}T^2 = T^*TT^2$ (3.2)

$T(TT^{*2}T^2) = (T^*T)T^2 \Rightarrow T(T^*TT) = (T^*T)T^2$ by (3.1). $(TT^* - T^*T)T^2 = 0$ or

$T^{*2}(TT^* - T^*T) = 0$. Since $\ker(T^*) \subset \ker(T)$, $TT^*(TT^* - T^*T) = 0$ and $\ker|T^*|^2 = \ker T^*$ implies

$$T^*(TT^* - T^*T) = 0 \tag{3.3}$$

$(TT^* - T^*T)T = 0$. Hence T is quasinormal . If $\ker(T^*) = 0$ then from (3.3) we obtain T is normal.

Theorem 3.2: If T and $T - I$ are in $[2QI] \cap [3QI]$, then T is quasinormal.

Proof: Since $T \in [2QI]$ and $T - I \in [2QI]$,

$$TT^{*2}T^2 = T^*TT \tag{3.4}$$

$$(T - I)(T - I)^{*2}(T - I)^2 = (T - I)^*(T - I)(T - I) \tag{3.5}$$

To prove that T is quasinormal, by part 2 of Theorem 3.1, it is enough to prove the kernel condition $\ker(T^*) \subset \ker(T)$ Since $(T - I) \in [3QI]$, $(T - I)^2(T - I)^{*2}(T - I)^2 = (T - I)^*(T - I)(T - I)^2$

$(T - I)(T - I)^*(T - I)^2 = (T - I)^*(T - I)(T - I)^2$ by (3.5).

On simplifying we obtain, $TT^*T^2 + TT^* - 2TT^*T + 2T^*T^2 - T^*T^3 - T^*T = 0$.

$TT^*T^2 + TT^* - 2TT^*T + 2(TT^{*2}T^2) - (TT^{*2}T^2)T - T^*T = 0$ by (3.4)

or $T^{*2}TT^* + TT^* - 2T^*TT^* + 2T^{*2}T^2\,T^* - T^*T^{*2}T^2T^* - T^*T = 0$.

Let $x \in \ker(T^*)$, then $T^*x = 0$. From the above equation, $-T^*Tx = 0 \Rightarrow Tx = 0$. Therefore $\ker(T^*) \subset \ker(T)$ and hence T is quasinormal.

Theorem 3.3: If $T \in [QI] \cap [nQI]$ then T^{n-1} is quasinormal.

Proof: By hypotheses given in the theorem, we have

$$T^{*2}T^2 = T^*T \tag{3.6}$$

$$T^{n-1}T^{*2}T^2 = T^*TT^{n-1} \tag{3.7}$$

We need to prove $T^{n-1}(T^{*n-1}T^{n-1}) = (T^{*n-1}T^{n-1})T^{n-1}$

$T^{n-1}(T^{*n-1}T^{n-1}) = T^{n-1}T^{*n-3}(T^{*2}T^2)T^{n-3} = T^{n-1}T^{*n-2}T^{n-2}$ by (3.6).

Repeated application of (3.6) gives, $T^{n-1}(T^{*n-1}T^{n-1}) = T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$ by (3.7)

$= T^{*2}T^2T^{n-1}$ by (3.6)

$$= T^{*3}T^{3}T^{n-1} \text{ by (3.6)}$$

$$= T^{*4}T^{4}T^{n-1} \text{ by (3.6)}$$

Repeating the process and using (3.6) we obtain the desired result.

4. CONDITIONS IMPLYING PARTIAL ISOMETRY

In this section we show that by imposing certain conditions on $[nQI]$ operator it becomes partial isometry.

Lemma 4.1: Let $T \in [nQI]$ then $T \in [(n+1)QI]$ if and only if $[T^*T^n, T] = 0$ where [A,B] = AB – BA.

Proof: $T \in [(n+1)QI] \Leftrightarrow T^nT^{*2}T^2 = T^*TT^n \Leftrightarrow T(T^{n-1}T^{*2}T^2) = (T^*T^n)T \Leftrightarrow T(T^*TT^{n-1}) = (T^*T^n)T$

$\Leftrightarrow T(T^*T^n) = (T^*T^n)T \Leftrightarrow [T^*T^n, T] = 0$.

Theorem 4.2: Let $T \in [(n+1)QI] \cap [nQI]$ such that T^n has dense range in H, then T is normal partial isometry.

Proof: By Lemma 4.1, $TT^*T^n = T^*T^n T$ or $(TT^* - T^*T)T^n = 0$. Since T^n has dense range in H, T is normal. Hence $(T^*T)^2T^n = T^{*2}T^2T^n = T^nT^{*2}T^2 = T^*T T^n$. Thus $[(T^*T)^2 - T^*T]T^n = 0$ on range of T^n and we have T^*T is a projection and hence T is a partial isometry by [2.2.1 Theorem 3[2]].

Corollary 4.3: If $T \in [(n+1)QI] \cap [nQI]$ such that T^n has dense range in H, then T is unitary.

Proof: By Theorem 4.2, T is normal and partial isometry and hence $TT^*T = T$. By the definition of $[nQI]$ $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$ or $T(T^{n-1}T^{*2}T^2) = T(T^*T^n)$ or $T^nT^{*2}T^2 = TT^*T^n$. (4.1)

Since $T \in [(n+1)QI]$, $T^*T T^n = T^nT^{*2}T^2 = TT^*T^n$ by (4.1). That is $T^*T T^n = TT^*T^n$. Using $TT^*T = T$,

we obtain, $T^*T T^n = TT^{n-1}$ or $(T^*T - I)T^n = 0$. Since range of T^n is dense in H, $T^*T = I$ and hence T is unitary.

Theorem 4.4: Let $T = UP$ be the polar decomposition of T and $T \in [2QI]$ such that $\ker(U) \subset \ker(U^*)$ then T is partial isometry.

Proof: $T \in [2QI]$ implies $TT^{*2}T^2 = T^*TT$ or $UP^2U^*P^2UP = P^2UP$. Taking adjoint

$PU^*P^2UP^2U^* = PU^*P^2$. The kernel condition $\ker(U) = \ker(P)$ yields, $UU^*P^2UP^2U^* = UU^*P^2$ or

$U^*UU^*P^2UP^2U^* = U^*UU^*P^2$ or $U^*P^2UP^2U^* = U^*P^2$. Since $U^*UP^2 = P^2$ we have,

$U^*P^2UP^2 = U^*P^2U$, or $U^*P^2U(P^2 - I) = 0$ or $[PU(P^2 - I)]^*[PU(P^2 - I)] = 0$.

Using the fact that, $S^*S = 0 \Rightarrow S = 0$ for any operator S on H, we obtain $PU(P^2 - I) = 0$. Again

$\ker(U) = \ker(P)$ yields $U^2(P^2 - I) = 0$. By hypothesis, $\ker(U) \subset \ker(U^*)$ and hence,

$U^*U(P^2 - I) = 0$ or $P^2 = U^*U$. That is P^2 is a projection and P is a partial isometry by [2.2.1 Theorem 3[2]]. Hence $T = UP$ is a partial isometry.

Remark 4.5: The above Theorem raises the following question: Is a 2 - power quasi - isometry T a partial isometry if $\ker(U^*) \subset \ker(U)$.

Theorem 4.6: Let T be of class $[nQI]$ such that T is a partial isometry then T^2 is an isometry.

Proof: T is a partial isometry implies $TT^*T = T$ (4.2)

Since $T \in [nQI]$ $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$.

$T(T^{n-1}T^{*2}T^2) = T(T^*TT^{n-1}) \Rightarrow T^nT^{*2}T^2 = TT^{n-1}$ by (4.2).

$$T^n(T^{*2}T^2 - I) = 0 \qquad (4.3)$$

That is, $T^{*2}T^2 = I$ on $\ker(T^n)$. By (4.3) $(T^{*2}T^2 - I)T^{*n} = 0$ or $T^{*2}T^2 = I$ on $\ker(T^n)^{\perp}$. Thus $T^{*2}T^2 = I$ on $H = \ker(T^n) \oplus \ker(T^n)^{\perp}$ implies T^2 is isometry.

Definition 4.7: The spectral radius of $T \in B(H)$ is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

It is well known that for a quasinormal operator, $r(T) = \|T\|$ [2].

Theorem 4.8: If $T \in [QI] \cap [2QI]$,then $r(T) = 1$ where $r(T)$ is the spectral radius of T.

Proof: Since $T \in [QI] \cap [2QI]$, by 4 of Theorem 2.1, T is quasinormal and hence $r(T) = \|T\| = 1$.

5. CLASS $[QZ_\alpha^n]$ OPERATORS

A.Uchiyama and T.Yoshino [7] studied a class of opeators T satisfying

$$\left|TT^* - T^*T\right|^{\alpha} \le c_\alpha^2 (T - \lambda I)(T - \lambda I)^* \text{ where } \alpha > 0 \text{ and } \lambda \in C.$$

Analogously, we define a new class $[QZ_\alpha^n]$ based on the n- power quasi isometry class $[nQI]$. We define a class $[QZ_\alpha^n]$ of operators T satisfying the following hypothesis. $T \in [QZ_\alpha^n]$ if for some $\alpha \ge 1$ and $c_\alpha > 0$, $\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^{\alpha} \le c_\alpha^2 (T - \lambda I)^{*n}(T - \lambda I)^n$ for all $\lambda \in C$. Equivalently, for some $\alpha \ge 1$ and $c_\alpha > 0$ $\left\|\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^{\frac{\alpha}{2}} x\right\| \le c_\alpha \left\|(T - \lambda I)^n x\right\|$ for all $x \in H, \lambda \in C$. Also let $[QZ^n] = \bigcup_{\alpha \ge 1} [QZ_\alpha^n]$. We note that the class $[nQI] \subset$ class $[QZ^n]$.

Lemma 5.1: For each α, β such that $1 \le \alpha \le \beta$, we have $[QZ_\alpha^n] \subseteq [QZ_\beta^n]$.

Proof: $\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^\beta = \left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^{\frac{\alpha}{2}} \left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^{\beta-\alpha} \left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^{\frac{\alpha}{2}}$ $\le \left\|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right\|^{\beta-\alpha} \left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^\alpha \le \left(\left\|T^{n+3}\right\| + \left\|T^{n+1}\right\|\right)^{\beta-\alpha} c_\alpha^2 (T-\lambda I)^{*n}(T-\lambda I)^n =$ $c_\beta^2 (T-\lambda I)^{*n}(T-\lambda I)^n$, where $c_\beta^2 = \left(\left\|T^{n+3}\right\| + \left\|T^{n+1}\right\|\right)^{\beta-\alpha} c_\alpha^2$. Therefore $[QZ_\alpha^n] \subseteq [QZ_\beta^n]$.

Proposition 5.2 [Berberian Technique [1]]

Let H be a complex Hilbert space. Then there exists a Hilbert space $K \supset H$ and an isometric $^*-$ homomorphism preserving the order $\Phi : B(H) \to B(K) : T \to T_0$ satisfying:

(1) $\Phi(T^*) = \Phi(T)^*$ (2) $\Phi(\lambda T + \mu S) = \lambda\Phi(T) + \mu\Phi(S)$

(3) $\Phi(I_H) = I_K$ (4) $\Phi(TS) = \Phi(T)\Phi(S)$

(5) $\|\Phi(T)\| = \|T\|$ (6) $\Phi(T) \le \Phi(S)$ if $T \le S$

(7) $\sigma(\Phi(T)) = \sigma(T), \sigma_a(T) = \sigma_a(\Phi(T)) = \sigma_p(\Phi(T))$

(8) If T is a positive operator, then $\Phi(T^\alpha) = |\Phi(T)|^\alpha \ \forall \alpha > 0$.

Lemma 5.3: If $T \in class[nQI]$, then $\Phi(T) \in class[nQI]$.

Lemma 5.4: If $T \in [QZ^n]$ then $\Phi(T) \in [QZ^n]$.

Proof: Since $T \in [QZ^n]$, $\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^\alpha \le c_\alpha^2 (T-\lambda I)^{*n}(T-\lambda I)^n$ for all $\lambda \in C$, $\alpha \ge 1$ and $c_\alpha > 0$. From the properties of Φ it follows that, $\Phi\left(\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^\alpha\right) \le \Phi(c_\alpha^2 (T-\lambda I)^{*n}(T-\lambda I)^n)$ for all $\lambda \in C$. By condition 8 of Proposition 5.2, we get, $\Phi\left(\left|T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right|^\alpha\right) = \left|\Phi\left(T^{n-1}T^{*2}T^2 - T^*TT^{n-1}\right)\right|^\alpha$ for all $\alpha > 0$. Therefore $\left|\Phi(T)^{n-1}\Phi(T)^{*2}\Phi(T)^2 - \Phi(T)^*\Phi(T)\Phi(T)^{n-1}\right|^\alpha \le \Phi(c_\alpha^2 (T-\lambda I)^{*n}(T-\lambda I)^n)$ for all $\lambda \in C$. Therefore $\Phi(T) \in [QZ^n]$

Theorem 5.5: Let $T \in [QZ^1]$,

(1) If $\lambda \in \sigma_p(T)$, such that $|\lambda| = 1$ then $\bar{\lambda} \in \sigma_p(T^*)$, furthermore if $\lambda \ne \mu$ then E_λ (the proper subspace associated with λ) is orthogonal to E_μ.

(2) If $\lambda \in \sigma_a(T)$ then $\bar{\lambda} \in \sigma_a(T^*)$.

(3) $T^{*2}T^2 - T^*T$ is not invertible.

Proof: (1) $T \in [QZ^1]$, then $T \in [QZ^1_\alpha]$, for some $\alpha \geq 1$, and therefore there exists a positive constant c_α such that, $\left|T^{*2}T^2 - T^*T\right|^\alpha \leq c_\alpha^2 (T-\lambda I)^*(T-\lambda I)$ for $\lambda \in C$. As $Tx = \lambda x$ implies $\left|T^{*2}T^2 - T^*T\right|^{\frac{\alpha}{2}} x = 0$ and $(T^{*2}T^2 - T^*T)x = 0$. $\lambda^2 T^{*2}x - \lambda T^* x = 0 \Rightarrow \lambda T^{*2}x - T^*x = 0$. By hypothesis $|\lambda| = 1$ and hence $(T^* - \overline{\lambda})T^*x = 0$. To establish $\overline{\lambda} \in \sigma_p(T^*)$ we need to show that $T^*x \neq 0$. Suppose $T^*x = 0$, then $0 = \langle x, T^*x\rangle = \langle Tx, x\rangle = \lambda\langle x, x\rangle$. Since $x \neq 0$, we obtain $\lambda = 0$ which contradicts, $|\lambda| = 1$ and hence the desired result. Moreover if $\lambda \neq \mu$, then $\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle$

Therefore $\langle x, y\rangle = 0$.

(2) Let $\lambda \in \sigma_a(T)$ then from condition 7 of Proposition 5.2, we have $\sigma_a(T) = \sigma_a(\Phi(T)) = \sigma_p(\Phi(T))$.

Therefore $\lambda \in \sigma_p(\Phi(T))$ .By Lemma 5.4 and condition 1 of proposition 5.2, we obtain,

$\overline{\lambda} \in \sigma_p(\Phi(T)^*) = \sigma_p(\Phi(T^*))$.

(3) $T \in [QZ^1]$, then there exists an integer $p \geq 1$ and $c_p > 0$ such that,

$$\left\| \left|T^{*2}T^2 - T^*T\right|^{2p-1} x \right\| \leq c_p^2 \|(T-\lambda I)x\|^2 \text{ for all } x \in H, \lambda \in C.$$

It is known that $\sigma_a(T) \neq \phi$. If $\lambda \in \sigma_a(T)$, then there exists a normed sequence (x_m) in H such that

$\|(T-\lambda I)x_m\| \to 0$ as $m \to \infty$. Then $(T^{*2}T^2 - T^*T)x_m \to 0$ as $m \to \infty$. Therefore $T^{*2}T^2 - T^*T$ is not invertible.

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